LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

ARNOLD SOMMERFELD CENTER FOR THEORETICAL PHYSICS

Impurity screening by defect in a spin-1 chain

Hong-Hao Tu

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arXiv:2307.09519, arXiv:2508.13114

Quantum Spin Liquids: Experimental Enigmas & Theoretical Challenges
Budapest, October 10th, 2025

Outline

- Boundary states in Conformal Field Theory (CFT): Cardy’s construction (illustrated with a spin-1/2 chain example)

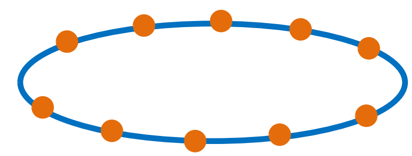
- Beyond Cardy: new boundary state in a spin-1 chain (and its generalizations)

- Summary & outlook

Boundary CFT in a nutshell: spin-1/2 chain example

- Spin-1/2 Heisenberg chain:

$$H = J \sum_{i=1}^{L} \vec{S}_i \cdot \vec{S}_{i+1}$$



  - ➢ Gapless, elementary excitations carry spin-1/2 (“spinons”)
  - ➢ Field theory: $\mathrm{SU}(2)_1$ Wess-Zumino-Witten (WZW) model

    Tomonaga-Luttinger liquid (with $K = 1/2$)

Central charge: $c = 1$

Primary fields: I & s

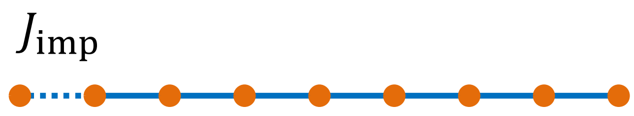
spin-0 spin-1/2

Bethe, Z. Phys. (1931); Faddeev & Takhtajan, Phys. Lett. A (1981);
Luther & Peschel, PRB (1975); Affleck & Haldane, PRB (1987); …

Boundary CFT in a nutshell: spin-1/2 chain example

- Kondo problem in an open spin-1/2 chain:

$$H = J_{\mathrm{imp}}\vec{S}_0 \cdot \vec{S}_1 + J\sum_{i=1}^{L-1}\vec{S}_i \cdot \vec{S}_{i+1}$$



  - ➢ $J_{\mathrm{imp}} = 0$: impurity is free ("free" boundary condition)

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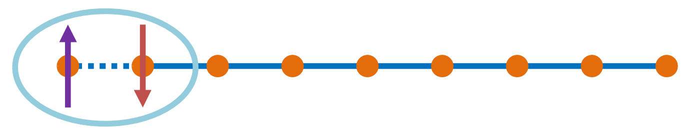

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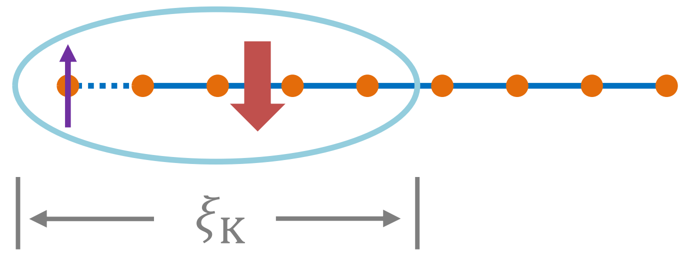


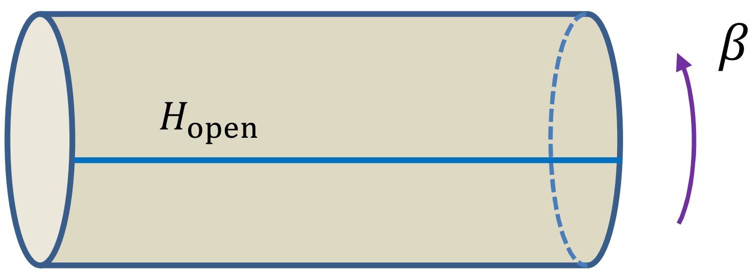
➢ $J_{\text{imp}} \to \infty$: impurity is screened ("screened spin" boundary condition")

$J_{\text{imp}} > 0$: screening cloud size $\xi_{\mathrm{K}} \sim e^{J/J_{\text{imp}}}$ (we consider $L \gg \xi_{\mathrm{K}}$)

Boundary CFT in a nutshell: spin-1/2 chain example

- Boundary CFT formulation ("loop-tree correspondence"):

$$Z = \mathrm{tr}(e^{-\beta H_{\mathrm{open}}})$$



Boundary CFT in a nutshell: spin-1/2 chain example

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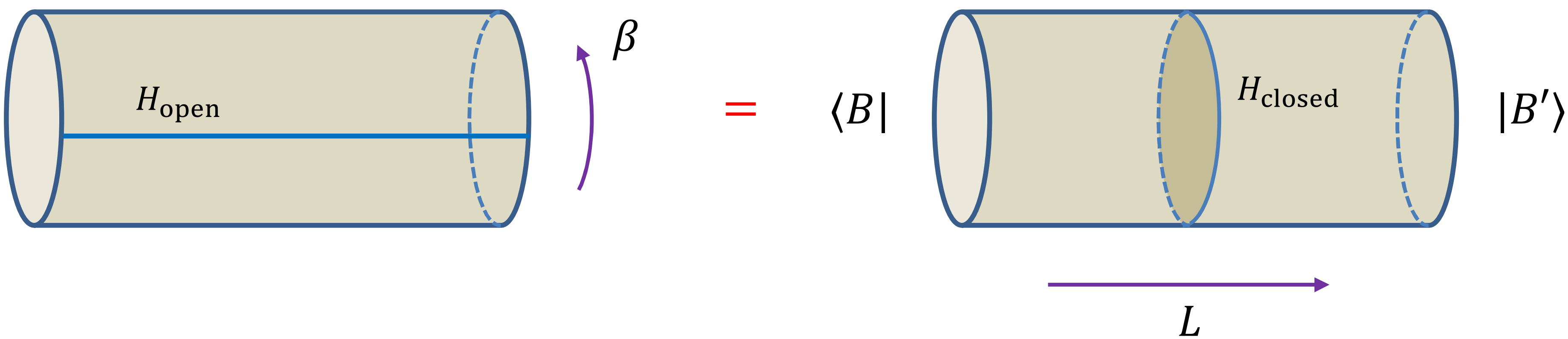
$$Z = \mathrm{tr}(e^{-\beta H_{\mathrm{open}}}) \;\; = \langle B | e^{-L H_{\mathrm{closed}}} | B' \rangle$$



$|B\rangle$ & $|B'\rangle$: boundary states

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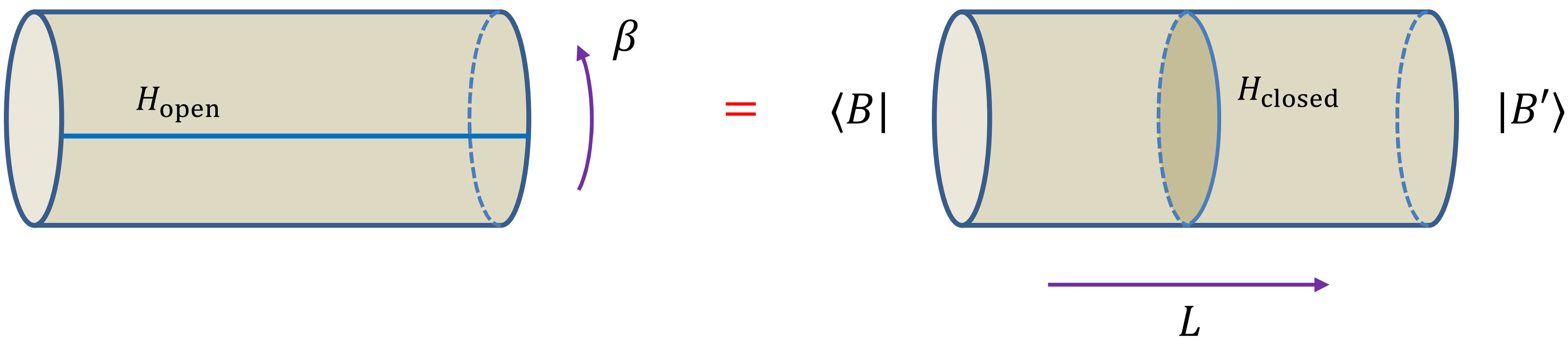

Cardy's solution: $$|B_a\rangle = \sum_b \frac{S_{ab}}{\sqrt{S_{0b}}} |b\rangle\rangle$$

Each primary field has a corresponding boundary state.

Cardy, NPB (1989).

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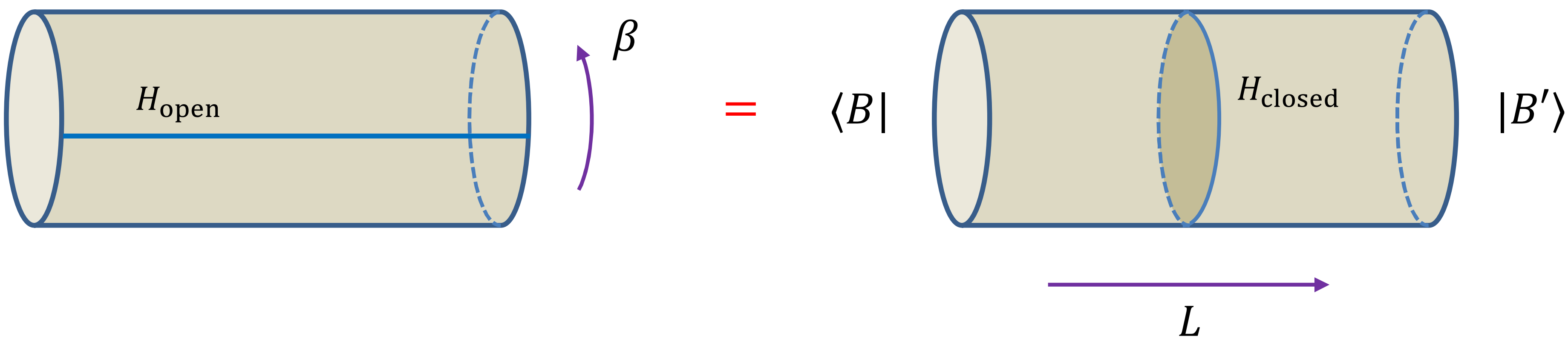

Cardy’s solution: $$|B_a\rangle = \sum_b \frac{S_{ab}}{\sqrt{S_{0b}}} |b\rangle\rangle$$

Caution: this does not provide a complete solution!

Non-Cardy example: “new” boundary state in three-state Potts model
Affleck, Oshikawa & Saleur, JPA (1998).

Boundary CFT in a nutshell: spin-1/2 chain example

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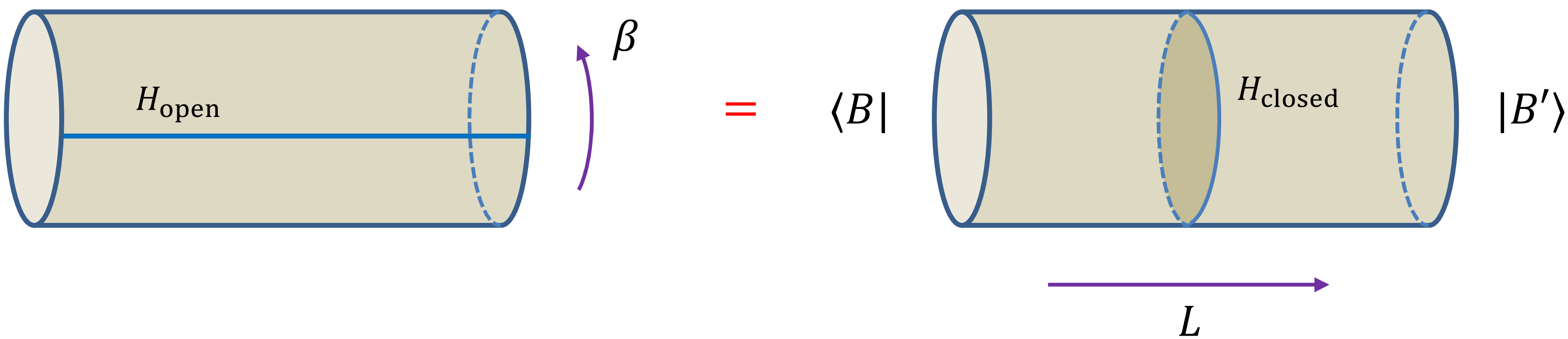
$$Z = \mathrm{tr}(e^{-\beta H_{\mathrm{open}}}) \;{\color{red}=}\; \langle B| e^{-L H_{\mathrm{closed}}} |B'\rangle$$



$L \to \infty$: $\quad Z = \underbrace{\langle B|0\rangle}_{g}\underbrace{\langle 0|B'\rangle}_{g'} e^{-L\varepsilon_0(\beta)}$

Affleck-Ludwig g-factor: universal quantity characterizing the boundary state

Affleck & Ludwig, PRL (1991).

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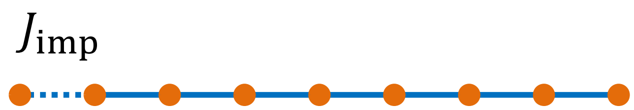

➢ $J_{\text{imp}} = 0$: impurity is free ("free" boundary condition)

$$|B_0\rangle = 2^{-1/4}(|0\rangle\rangle + |1/2\rangle\rangle)$$

➢ $J_{\text{imp}} \to \infty$: impurity is screened ("Kondo" boundary condition")

$$|B_{1/2}\rangle = 2^{-1/4}(|0\rangle\rangle - |1/2\rangle\rangle)$$

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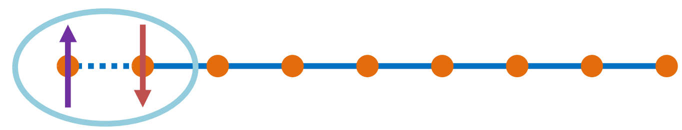
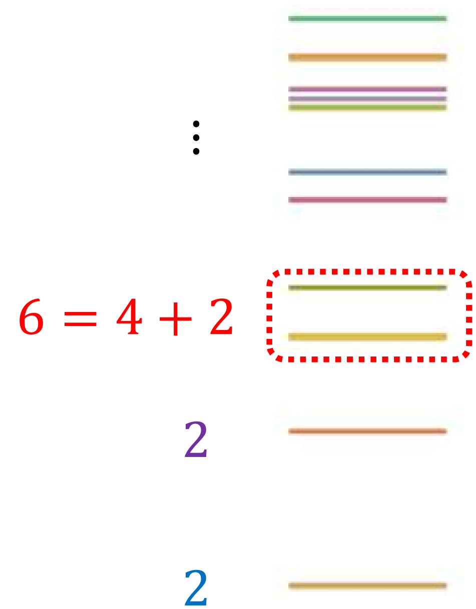


- Prediction of low-energy spectrum ($J_{\text{imp}} > 0$):

$$\begin{aligned} Z_{1/2,0} &= \langle B_{1/2} | e^{-LH_{\text{closed}}} | B_0 \rangle \\ &= \chi_{1/2}(q) \\ &= q^{1/12}(2 + 2q + 6q^2 + \cdots) \end{aligned}$$



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Impurity screening in a spin-1 chain

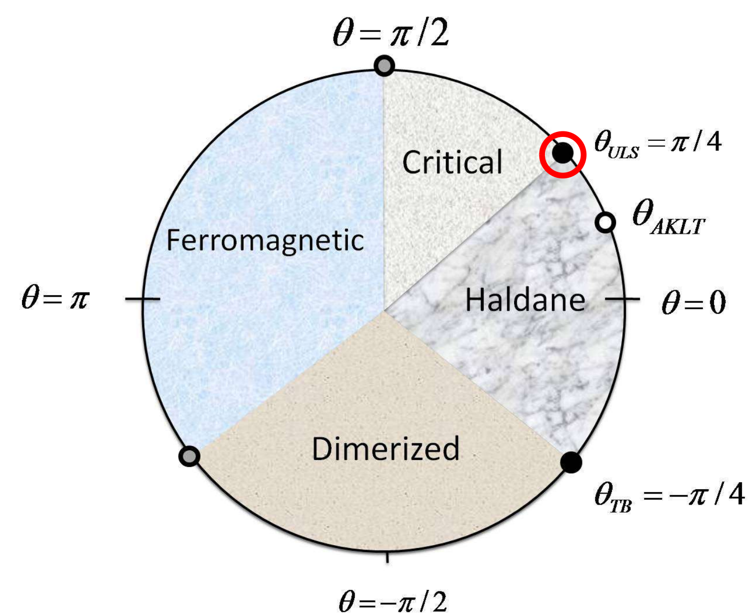
- Spin-1/2 impurity coupled to an open spin-1 chain:

$$H = J_{\mathrm{imp}} \vec{S}_0 \cdot \vec{T}_1 + H_{\mathrm{bath}}$$

$$H_{\mathrm{bath}} = \sum_{i=1}^{L-1} \cos\theta\,(\vec{T}_i \cdot \vec{T}_{i+1}) + \sin\theta\,\left(\vec{T}_i \cdot \vec{T}_{i+1}\right)^2$$

Uimin-Lai-Sutherland (ULS) point: $\theta = \pi/4$

- ➢ Gapless, elementary excitations carry spin-1
- ➢ Field theory: $SU(3)_1$ WZW



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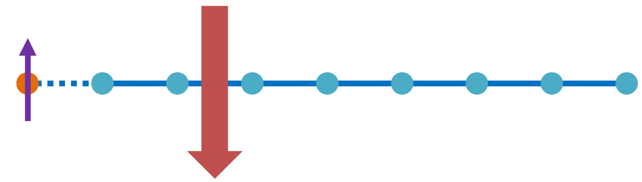

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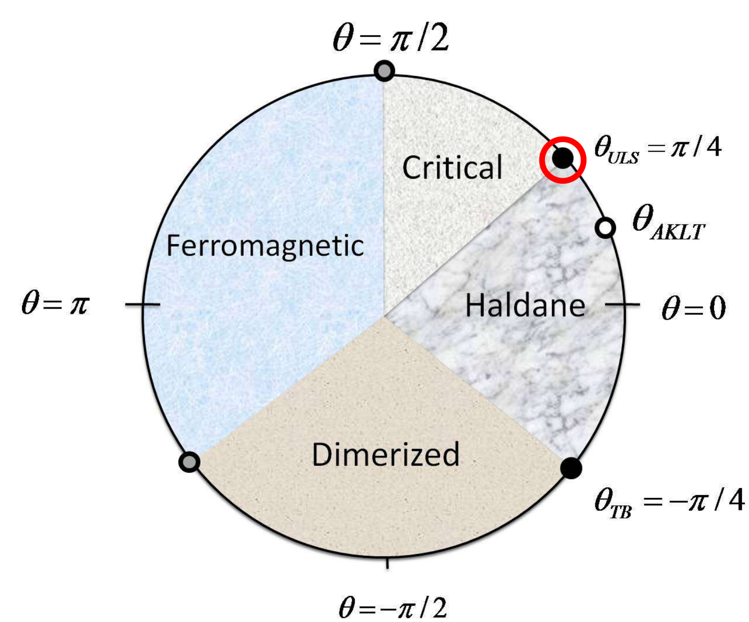

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Spin-1/2 impurity cannot be screened ?

Topological defect line in $\mathrm{SU}(3)_1$ CFT

- Conformal embedding: $\mathrm{SU}(2)_4 \subset \mathrm{SU}(3)_1$

Primary fields:			Same central charge: $c = 2$
	0	$\mathbf{0}$	
	1/2	$\mathbf{3}$	
	1	$\overline{\mathbf{3}}$	
	3/2		
	2		

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- 1/2 and 3/2 do not show up in the operator content of $\mathrm{SU}(3)_1$, but they can be used to construct a (non-Verlinde) "topological defect line" for $\mathrm{SU}(3)_1$:

$$\sigma = 1/2 \oplus 3/2$$

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$$\sigma = 1/2 \oplus 3/2 \quad \Rightarrow \quad |B_\sigma\rangle = \sigma |B_{\mathbf{0}}\rangle$$

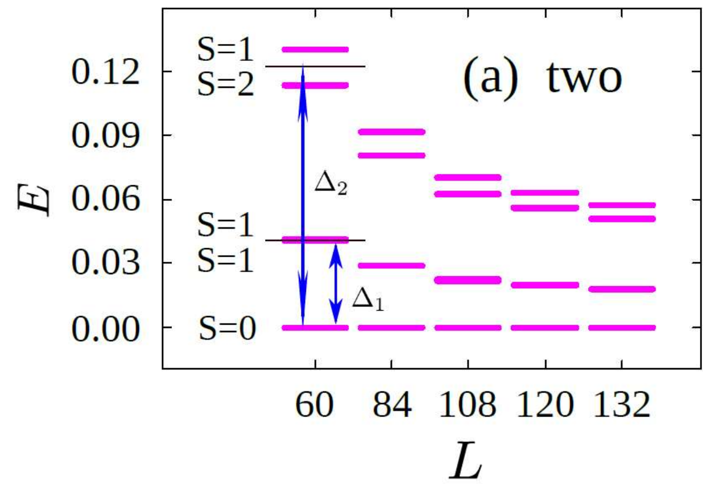
Non-Cardy boundary state!

Low-energy spectrum: field theory vs. numerics

- Two impurities:

$$Z_{\sigma,\sigma} = \langle B_\sigma | e^{-LH_{\text{closed}}} | B_\sigma \rangle$$

$$= q^{-1/12}(1 + 6q^{1/3} + 8q + \cdots)$$



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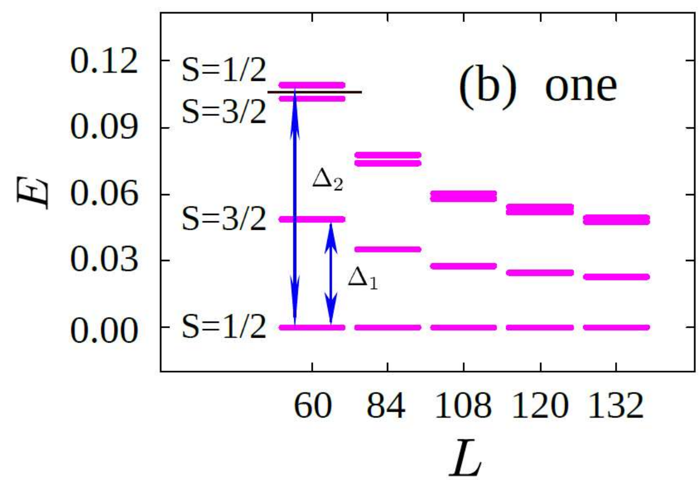

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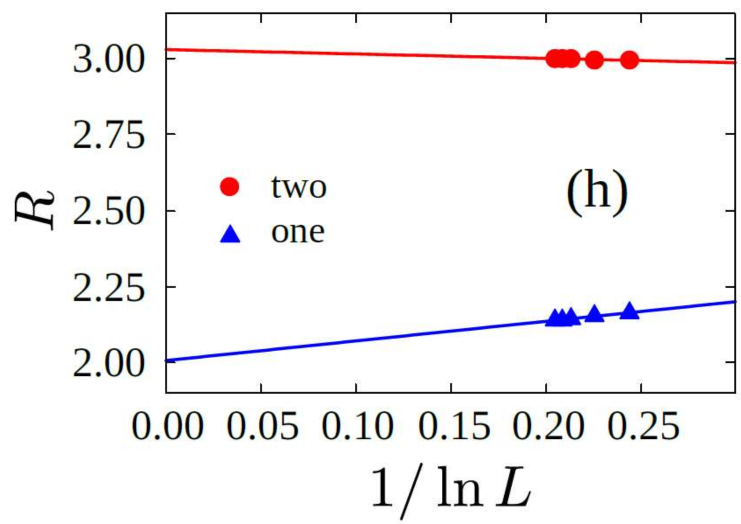
$R = \Delta_2/\Delta_1$



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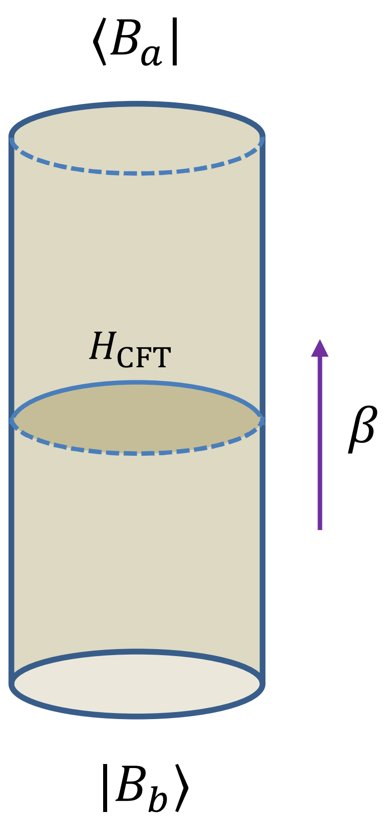
Lattice realization of boundary states

- Cardy's insight: gapped ground states in the vicinity of a critical point can be naturally identified with conformal boundary states.

$$Z_{ab} = \langle B_a | e^{-\beta H_{\mathrm{CFT}}} | B_b \rangle$$



Cardy, SciPost (2017).

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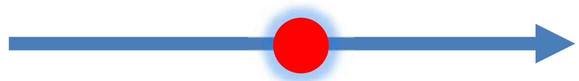
Example: transverse-field Ising chain

$$H = -\sum_{i=1}^{L} \sigma_i^z \sigma_{i+1}^z - h \sum_{i=1}^{L} \sigma_i^x$$



h

$h = 1$ (Ising CFT)

$$|B_I\rangle \to |\uparrow\uparrow\uparrow \cdots \uparrow\rangle$$

$$|B_\varepsilon\rangle \to |\downarrow\downarrow\downarrow \cdots \downarrow\rangle$$

$h = 0$

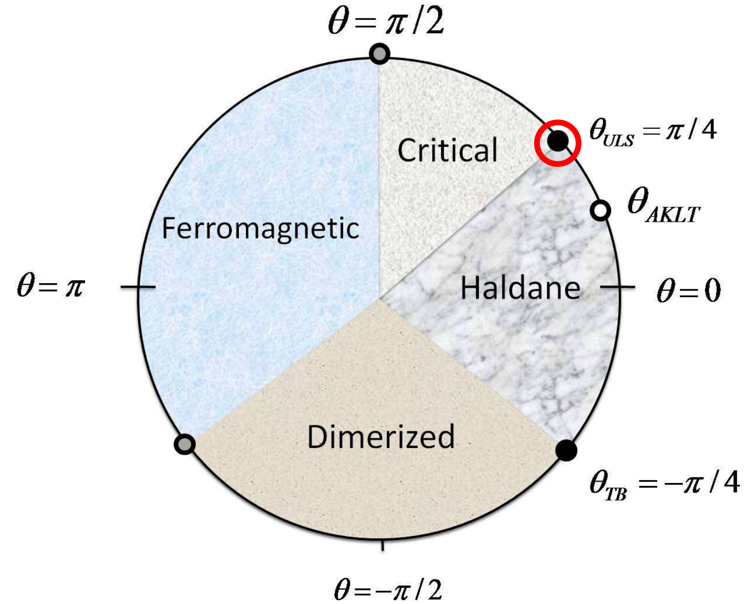
$$|B_\sigma\rangle \to |{+}{+}{+} \cdots {+}\rangle$$

$h \to \infty$

Lattice realization of boundary states

- Spin-1 ULS chain realizing $\mathrm{SU}(3)_1$ CFT:

$$H = \sum_{i=1}^{L} \cos\theta \, (\vec{T}_i \cdot \vec{T}_{i+1}) + \sin\theta \left(\vec{T}_i \cdot \vec{T}_{i+1}\right)^2$$



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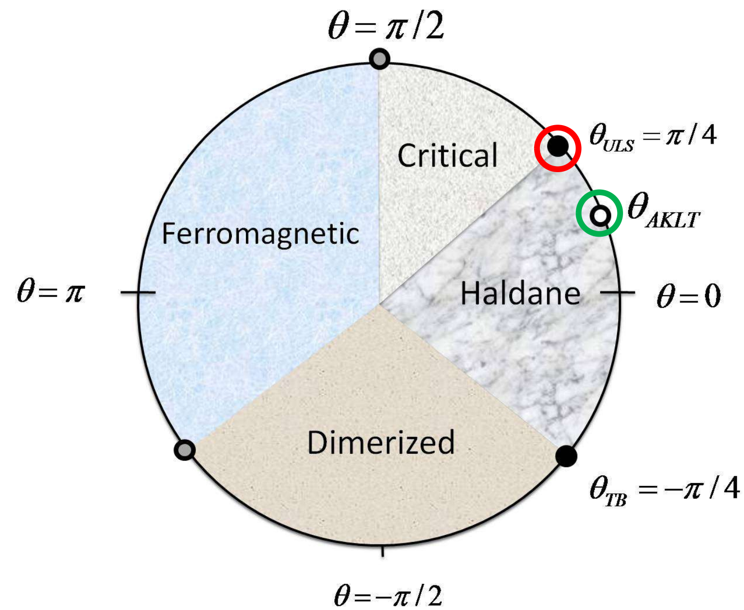

Lattice realization of boundary state $|B_\sigma\rangle$: AKLT state

verified by $\langle \mathrm{AKLT}|\mathrm{ULS}\rangle = g_\sigma e^{-\alpha L}$ with $g_\sigma = \sqrt[4]{3}$

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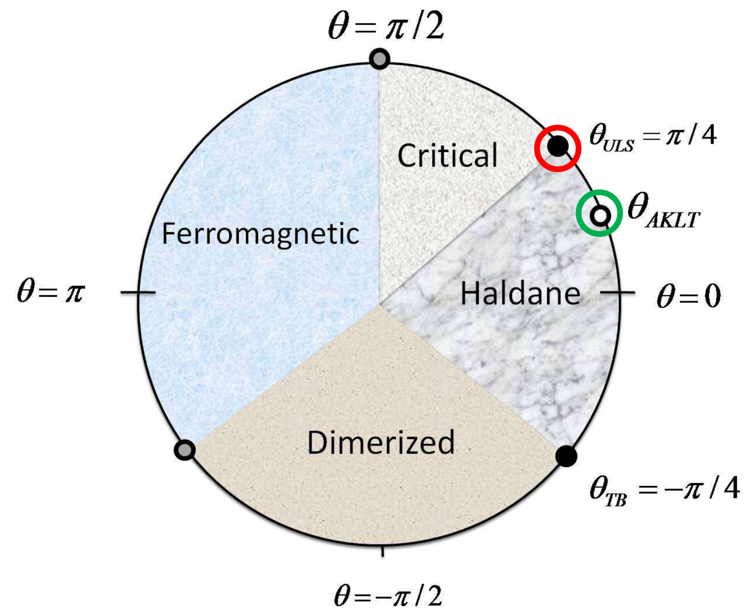

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Determinant formula exists (recent progress in integrability techniques)

Pozsgay, Piroli & Vernier, SciPost (2019);

Gombor, PRL (2025); …

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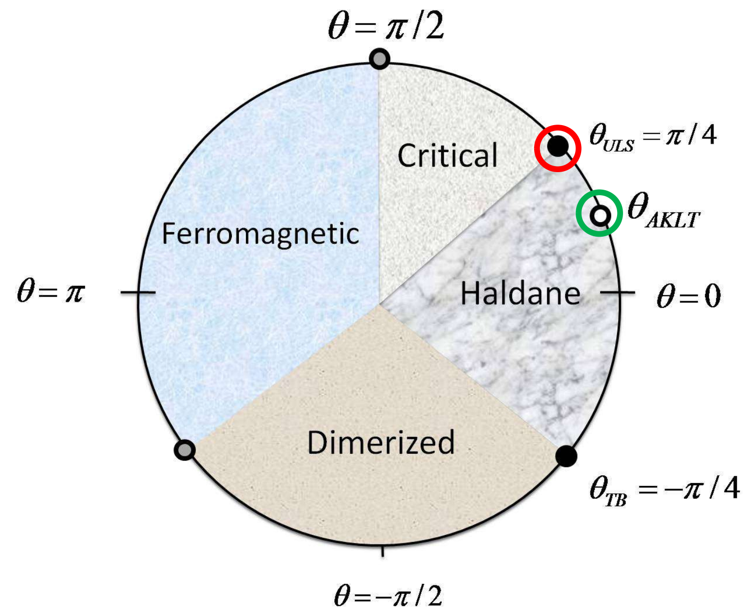

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Further examples: $\mathrm{SO}(n)_2 \subset \mathrm{SU}(n)_1$

SO(n) AKLT state is a non-Cardy boundary state of the SU(n) ULS chain!

HHT, G.-M. Zhang & T. Xiang, PRB (2008)

Summary & outlook

- We have demonstrated that conformal embedding provides a new mechanism for constructing a family of conformal boundary states beyond Cardy's.

- Cardy's insight hints at a deep connection between conformal boundary states and 1D symmetry-protected topological (SPT) states.
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Thank you for your attention!